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NUMBER SYSTEMS OF THE NORTH AMERICAN INDIANS.*

By W. C. EELLS, Whitworth College.

The linguistic diversity of the Indians inhabiting the North American continent is one of the most remarkable features of world ethnology.[1] The late director of the Bureau of American Ethnology[2] says: "In philology North America presents the richest field in the world, for here is found the greatest number of languages distributed among the greatest number of stocks." The Bureau recognizes almost three score distinct linguistic families having no lexical resemblance, no apparent unity of origin, no relation to European or Asiatic languages. These "families" are further subdivided linguistically into 750 "tribes" or languages.[3]

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[1] *Bibliographical Note.*—It is impossible to give a comprehensive bibliography of this subject in a small space. Most of the material is in hundreds of separate vocabularies, grammars, dictionaries and discussions of the various languages of the American Indians. Considerable may be found in the reports of the Bureau of American Ethnology, Washington, D. C. Pillings Bibliographies, published by this Bureau from 1887 to 1894, contain references to most of the literature up to the date of their publication for the nine most important linguistic families. In addition may be mentioned as especially important: Conant, L. L., *The Number Concept*, N. Y., 1896; Cushing, F. H., "Manual Concepts," in *Amer. Anthropologist*, 1892, p. 289; Eells, M., "Indians of Puget Sound: Measuring and Valuing," in *Amer. Antiquarian*, Vol. 10, p. 174; Dixon, R. B., and Kroeber, A. L., "Numeral Systems of the Languages of California," in *Amer. Anthropologist*, N.S., Vol. 9, p. 663; Powell, J. W., "Linguistic Families of America, North of Mexico," in Seventh Ann. Rpt., Bur. of Eth., 1885–86; Trumbull, J. H., "On Numerals in American Indian Languages," in *Trans. Amer. Philological Assn.*, 1874, p. 41. Full credit can scarcely be given for each statement made in this paper. The above mentioned sources have been used freely, but even more the numerous dictionaries and grammars mentioned at the beginning of this note. A bibliography of about 300 titles prepared by the author is on file in the library of the University of Chicago, the Newberry Library, Chicago, and the library of the University of Wisconsin.

[2] Major J. W. Powell: 1st Ann. Rpt. Bur. Eth., 1879–80, p. 78.

[3] For a list of these families and tribes see Powell: 7th Ann. Rpt. Bur. Eth., p. 1; Hodge: *Handbook of American Indians*, article "Linguistic Families." Not all these languages were entirely distinct, but most of them were. They are analogous to the French, Spanish, and Italian as different "languages" of the same "family." It is a conservative statement to say that about 500 distinct languages have been spoken on the American continent.

These languages differ as widely in *number words* and *number syst ms* as they do in other features. This is in marked contrast with the languages of the great Indo-European family where, even in languages which are mutually unintelligible, the same root words appear with great uniformity in the numerals. The very remarkable differences in the form and use of the numerals of the American Indians afford a fruitful field for study of the evolution of the concept of number among hundreds of distinct, uncivilized peoples. This paper is based upon an examination of the number systems of more than three hundred of these languages in North America.[4] will discuss the origin of number words and their principles of formation, the way in which they were built up into number systems, and some of the variations of these systems in actual use.[5]

I. Principles of Formation.

1. Digital Origin. The child's most natural counters are his fingers; to them he turns almost instinctively when wishing to count. What evidence is there that primitive peoples, races in the childhood state, have also turned to their digits for assistance? The answer to this question will throw much light on the origin of number words and their development into systems. We shall consider three kinds of evidence.

(*a*) *Evidence from systems used.* The almost universal prevalence of decimal, quinary, or vigesimal systems of numeration on the North American continent is perhaps the strongest general evidence that counting in its origin is digital. But the octonary, quaternary and ternary systems mentioned later will show that such evidence is neither universal nor conclusive. To this indirect evidence, based on the systems used, can be added direct proof from observation and from the ascertained meaning of number words.

(*b*) *Observational evidence.* Many observers report that Indians in various parts of the continent use their fingers, or fingers and toes, in counting, at the

[1] *Nature of Sources.*—Before the arrival of the white man the Indian had no written language, except a system of rude hieroglyphics among some of the more intelligent tribes. Reproductions of many of these are given by Mallery in the 4th and 10th Reports of the Bur. of Eth. They have but little of mathematical interest. The oral number systems of the Indians are the important sources available. But these have been committed to writing by hundreds of different men, of varying reliability and familiarity with the languages, of different nationalities, using various systems of spelling, at different dates, and extending to various limits. Aside from this is the fact that the same Indians are often known by a dozen or twenty different names, or the same name is applied to several distinct tribes speaking different languages. Such confusion in the sources makes it extremely difficult to classify them satisfactorily for the purposes of comparative study. Many errors in detail must have been made which can only be corrected by further study and reference to experts in American linguistics in various parts of the country. But it is hoped that such errors as have been made are not serious enough to affect much, if at all, the general results and statements made in this paper.

[2] *Nature of Conclusions.*—The conclusions stated in what follows are usually based entirely on the relative *number* of languages of the 324 examined. Little effort has been made to indicate the amount of territory covered or the number of Indians involved. Some of the languages were used by only a few people, others by many thousands. But for the purposes of this paper a small tribe with peculiarities in its number system, is as interesting and as important as a much larger one.

same time speaking the corresponding number words. With some tribes the *use* of the fingers is the important thing, the accompanying vocal utterance being of secondary importance; e. g., some of the Eskimo tribes use the same *words* for 6, 7, 8, 9, 10 as for 1, 2, 3, 4, 5 but count them on the second hand. In others the language development has led to independent numerals which often preserve evidence of their digital origin. Examples are given in the next section. In widely separated tribes all over the continent actual finger counting has been observed and the rather remarkable fact noted that the order of counting is almost always uniform, commencing with the little finger of one hand and counting to the thumb, thence to the thumb of the other hand and to the little finger again. Usually the fingers are bent as the counting continues, but sometimes the hand is first clenched and the fingers then extended one at a time.[1] This general uniformity of order gives considerable aid in the linguistic discussion of the next paragraph.

(*c*) *Linguistic evidence.* The German philologist Grimm in speaking of the Old World languages says: "Alle Zahlwörter gehen aus von den Fingern der Hände."[2] Is this observation true for the New World? In the languages of civilized nations the numerals are so ground down from long usage that it is difficult to detect in their form their possible digital origin. While this is also the case in some Indian languages, in many there are striking similarities, while in others digital words and number words are almost or quite identical. Following the observed order of counting, the little fingers would be used for 1 and 10, the fourth fingers for 2 and 9, etc. Only a few typical examples of the many noted will be given with reference to each number.

The number ONE. Some of the names given the little finger are "the smallest," "the last of the hand," "little daughter of the hand." While we do not expect to find the word for *one* always or even usually connected with that for little finger (since the concept of unity doubtless preceded formal counting) yet some instances are known; e. g., Massachusetts: *pasuk* from *piasuk*, "very small"; Montagnais: *inlare*, "end is bent"; Zuni: *topinte*, "taken to start with." Two is sometimes derived from finger; e. g., Montagnais: *nake*, "another bent in"; Dakota: *nonpa*, "to bend down"; Zuni: *kwillin*, "that (finger) put down with its like." But more often it is connected with the word for "hand," probably because there are two hands.[3] THREE. The third finger is often named "the middle"; e. g., Massachusetts: *nishwe*, from *nashaue*, "half way"; Zuni: *hain*, "equally dividing one." FIVE is counted on the thumb and we have Karankawan: *natsa behema*, from *natsa*, "one," *behema*, "finger." But more prominent is the idea that the hand is completed, variously expressed as "finished," "fingers finished," "all fingers," "all done," etc.; e. g., Ojibwa: *nanan*,

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[1] Is this order the natural one? Many children have been observed to use their thumb first in counting. But Cantor has noted that the South African savages use the same order as the Indian order given above. See Cantor: *Vorlesungen über Geschichte der Mathematik*, Band I, Dritte Auflage, Leipzig, 1907, pp. 6–7.

[2] Grimm: *Geschichte der deutschen Sprache*, I, p. 167.

[3] Discussed more fully below under "Duplicative Principle."

"gone," "spent," and similarly in several Eastern languages. Hidatsa: *kichu*, from *ki*, "completely," *chu*, "turned down"; Ute: *munugi*, from *manoku*, "all." In most instances however it is connected with "hand," or "whole hand"; e. g., Kaniagmiut: *talgamen*, from *talega*, "hand"; Comanche: *mowaka*, from *mowa*, "hand"; Klamath: *tunep*, from *tu*, "away," *nep*, "hand," i. e., "hand-away."

From six to nine the numerals are expressed (*a*) from the names of the fingers used; (*b*) by "hand" + 1, 2, 3, 4; (*c*) by 1, 2, 3, 4, + "again" or "besides" (most frequent); or (*d*) by 1, 2, 3, 4 repeated without change. This last method is found only among the Eskimo and is probably always accompanied by the actual use of the second hand.

SIX. The Point Barrow six well illustrates the evolution of a number word. We are given the three forms

atautyimin-akbinigin-tudlimut, literally,	"once-on next- (and) five,"
atautyimin-akbinigin	"once-on next,"
akbinigin	"on next."

Other examples of six are Tano: *manli*, from *man*, "hand," *li*, "piece," i. e., "hand and piece of next"; Klamath: *nadshk-shapta*, "one I have bent over"; Takelma: *maimis*, "finger one in." SEVEN falls on the index finger or "pointer," e. g., Zuni: *tserucek* from *tserwerc*, "to point"; Greenland: *arfinek-mardluk*, "on the other hand-two"; Omaha: *penompa*, from *pe*, "finger," *nompa*, "two." For EIGHT, Hudson's Bay: *kittukleemot*, "middle finger"; Omaha: *pethatbathi*, "finger-three"; Klamath: *ndan-kshapta*, "three I have bent over." For NINE the subtractive principle comes into use and we have the additional forms "one left," "only one," etc.[1] We also have the forms, Greenland: *mikkelerak*, "fourth finger"; Zuni: *tenalikya*, "all but one held up with rest." TEN is counted on the little finger, e. g., Hudson's Bay: *eerkitkoka*, "little finger." But more prominent is the fact "two hands completed," "man finished," or "man." Thus Zuni: *astemthla*, "all of the fingers"; Wintun: *pampa-sempta* from *pampu-ta*, "two," *sem*, "hand"; Konkau: *machoko*, from *mar*, "hand," *choko*, "double."

Above ten, various combinations of the first ten numerals occur in which of course these digital names reappear. A few other examples of interest will be given. ELEVEN. Unalit: *atkhakhtok*, "it goes down" (referring to change from hands to feet). THIRTEEN. Greenland: *arkanenpingasut*, "on the first foot, three," etc. SIXTEEN. Unalit: *gukhtok*, "it goes over" (to toes of other foot). NINETEEN. Maidu: *tsoi-ni-maiduk*, from *maidu*, "man," *tsoi*, "four"—"four with man," i. e., after 15, 4 on toward 20 (man); and similarly for 16, 17, 18. TWENTY. In decimal-system languages twenty is usually but not always "two tens." In the vigesimal it is quite commonly "man," "Indian," "all hands and feet"; e. g., Navaho: *natin*, from *tine*, "man"; Greenland: *inuk-navdlugo*, "man come to an end," or *inup-avatai-navdlugit*, "man's outer members completed"; Kaniagmiut: *swinuk*, from *suk*, *innuk*, "man"; Wintun: *ketet-wintun*, from *ketet*, "one,"

[1] See examples below under "Subtractive Principle."

wintun, "Indian"; Tuolomne: *renge mewoom,* "one man"; Maidu: *kom maiduk,* from *maidu,* "Indian," and *kom,* possibly "whole"; Shasta: *tsec,* from *tsec,* "man"; Tlingit: *tlekha,* from *tle,* "one," *hka,* "man."

(*d*) *Extent and distribution.* Clear linguistic digital evidence similar to the examples given above has been found in about 40 per cent. of the languages examined, uniformly distributed over the continent. Doubtless further study will reveal similar evidence in other languages especially where adequate vocabularies have not been available.

(*e*) *Non-digital evidence.* We turn now to a consideration of the evidence that the origin of number words was non-digital in some languages. There are four phases to be considered. (1) *First Four Numerals.* The concepts of unity and duality are so fundamental that in many instances we may be sure they were named before formal finger counting gave names to the corresponding words. *One* has a connection with the first personal pronoun in some languages. *Two* seems often to come from roots denoting separation, "that" as distinguished from "this," or from ideas of pairs, being frequently related to the words for hands, feet, eyes, wings, husband and wife. *Three* is more frequently digital, but it seems sometimes to have a meaning of "more," "many," a plural as distinguished from a dual. Compare Micmac: *tchicht,* "three," with the cognate Delaware *tchitch,*" still more." *Four* is sometimes expressed by a word meaning "complete," "right," "perfect." Its frequency as a sacred number among the North American Indians and its use in some cases as the base of a quaternary system[1] indicate that it is a unique word of non-digital origin. (2) *Arithmetical Operations.* Numbers higher than ten and in many cases those higher than five are expressed by arithmetical operations, and the digital meaning, even if present in the beginning, usually sinks into the background. The process of such combination begins earlier than the English in many Indian languages. We have numerous examples of $3 = 2 + 1$, $4 = 2 \times 2$, $4 = 2 + 2$, $6 = 3 \times 2$, $8 = 4 \times 2$, $10 = 5 \times 2$, $12 = 6 \times 2$, $9 = 3 \times 3$ and other rarer combinations. Thus there are many cases in which words for numerals above three are derived by purely arithmetical processes.[2] Of course there are the higher numerals, hundred, thousand, million, where they exist, in which we should rarely expect digital evidence. (3) *Marks of Completion.* When the Indian has counted ten or twenty he may use some reminder of the fact, such as a pebble, stick, arrow, grain of corn, etc. For example, Huchnom: 20, *pualya,* "one-stick-stand" and similarly for 40 and 60; in the same language 100 is *pual,* "one-stick" and similarly for 200; Maidu: 20, *penim nokom* "two arrow"; Gallinomero: 100, *tcacuto-hai,* "ten-stick." (4) *Superlative and Indefinite.* When a simple arithmetical combination is not used, especially for the expression of higher units, a superlative principle is sometimes found. Hundred is often expressed, "big ten" and thousand as "old hundred," "big hundred" or "too many to count"; e. g., Delaware: 1,000, *ngutti kittapachki,* "the great hundred"; Choctaw:

[1] See below under "Quaternary Systems."

[2] See examples below under "Duplicative Principle," "Ternary Systems."

1,000, *tahlepa siponki,* "old hundred"; Kwakiutl: 1,000,000 *tlinhi,* "number which cannot be counted"; compare the Greek "myriad."

We conclude that in North American Indian languages it is by no means true that number words, even as far as ten, always "gehen aus von den Fingern" although they probably do in a large majority of cases and the close connection can be traced in many instances. There is little uniformity as to method of formation, considerable diversity being found even in adjacent languages of the same family. This would indicate that their separation into tribes preceded the development of formal counting.

2. **Additive Principle.** Cantor says that addition and multiplication are two methods of counting as old as the formation of number words.[1] The additive principle is found of course in all numeral systems. Three phases of it are of interest in the American Indian languages.

(*a*) *Repetition.* This is the simplest form of the additive principle. If "one" is given, either as a symbol or as a word, "two" may be expressed "one-one," "three" as "one-one-one," etc., or by symbols as in the Roman numerals from one to four. In the gesture language of the Indians this is the method used, the fingers being the counters. In spoken language no instance has been found of "two" as "one-one," but there are several of "four" as "two-two"; e. g., Catawba: 2, *purra,* 4, *purrapurra.* In the Indian pictographs or hieroglyphics the simple repetition of strokes or notches is used, even for numbers up to a hundred. Sometimes these are grouped into tens by longer strokes or larger notches.

(*b*) *Addition to a Base.* The English does not begin to use the additive principle until ten is reached, but many Indian languages begin much earlier. The earliest instance found is in the Coahuiltecan: 1, *pil,* 2, *ajtic,* 3, *ajtic-pil.* In other languages we find such expressions as "6–2 added" for 8, "$8 + 1$" for 9, "$12 + 3$" for 15, and of course very often "$5 + 1, 2, 3, 4$," for 6, 7, 8, 9 and "$10 + 1, 2, \cdots 9$" as in English for 11, 12 $\cdots$ 19. Those from 15 to 19 are also represented by "$15 + 1, 2, 3, 4$." An interesting variation is shown by the Maidu numerals from 16 to 19 which in translation are "one with man" for 16, "two with man" for 17 and similarly for 18 and 19 to 20 "man," the thought being "15 and one more on toward entire man."

(*c*) *Precedence.* Hankel and Fink call attention to a general law by which the written representation of numbers, when not confined to the mere rudiments, shows a tendency for higher numerals to precede the lower to represent addition.[2] Is there a similar tendency for the spoken order of numerals among the Indian languages? Does the lower precede the higher or vice versa? In about 250 languages sufficient facts were available for study of the method of formation of compound numerals by the additive principle. The groups from 5 to 10, from 10 to 20, and above 20 have been considered separately. Using the notation "$G + L$" to indicate "the greater is followed by the less" and "$L + G$" for the

[1] Cantor: *op. cit.*, p. 8.

[2] Hankel, H.: *Zur Geschichte der Mathematik in Alterthum und Mittelalter*, Leipzig, 1874, p. 32; Fink, K.: *History of Mathematics* (Beman and Smith trans.), Chicago, 1900, p. 8.

opposite, "$5 + 1$," "$1 + 5$," etc., to stand for pure number combinations in the order given, and "$1 + X$," "$X + 1$" for the indefinite cases of an unknown element (probably non-numerical such as "again," "besides," etc.) combined with the known numbers, we may summarize the results of an examination of these languages as follows.

In the 5–10 group for the pure number combinations, $L + G$ and $G + L$ occur with about equal frequency. But if we include the unknown compounds $X + 1$ and $1 + X$ with $G + L$ and $L + G$ respectively, $L + G$ predominates about $2 : 1$. In the 10–20 group for pure number combinations $G + L$ predominates strongly, $8 : 1$, a reversal of the order of the first group. But for $X + 1$ and $1 + X$, $L + G$ predominates slightly, the ratio being $4 : 3$, so that $G + L$ predominates in the group $2 : 1$. In the group of higher combinations $G + L$ predominates $16 : 1$. Combining these results $G + L$ predominates altogether about $2 : 1$. If the indefinite cases of $1 + X$ and $X + 1$ are excluded, only pure number combinations remaining, we approach close to a definite law. $G + L$ then predominates about $8 : 1$. $G + L$ and $L + G$ both occur in the same language in fully half the cases examined. In this particular a marked contrast with the multiplicative principle will be observed. It may be mentioned that our own (oral) system is mixed, $L + G$ from 10 to 20, and $G + L$ for higher numbers.

3. **Subtractive Principle.** Fink says: "In the verbal formation of a number system very rarely does subtraction come into use."[1] This statement is scarcely warranted for the systems of the American Indians for the principle has been found in 40 per cent. of the languages examined. It occurs most frequently in expressions for "nine"; e. g., "one finger left," "one from ten," "one from finished," etc., but also in the cases of 4, 7, 8, 14, 17, 18, 19 and the odd tens, 30, 50, etc. It is widely but not uniformly distributed over the continent It occurs most frequently in the northern, eastern and central sections of the continent, less on the Pacific coast and least of all about the Gulf of Mexico. As is to be expected, "one subtracted" is most usual, occurring in about 30 per cent. of the languages, "two subtracted" in about 5 per cent. and "three subtracted" and "ten subtracted" in about 2 per cent. each. The use of the principle is found in some languages but not in closely related ones of the same family.

(*a*) *One Subtracted.* As examples we may give for *nine,* Unalit: *keka-mitatet,* "nearly ten," or *payuk-ostau,* from *payuk,* "one," *ostau,* "less?"; Uinta: *suromatampsuin,* "near ten"; Haida: *klath-skwanson,* from *klath,* "one," and *skwansin,* "ten." For *four* we have Takhtam: *voatcham,* from *mah-atcham,* "five"; Zuni: *awiten,* "all fingers all but done." For *fourteen,* Point Barrow: *akimiaxotaityuna,* "I have not quite 15." For *nineteen,* Alaska: *enuenok-otalia,* "twenty less one." *Thirty-nine,* Kulanapo: *pitikunanu-akhokaki,* "forty, one not." Arikara expresses both 7 and 9 from 8 and 10 respectively by means of a diminutive particle, "little ten," "little eight."

[1] Fink: *op. cit.,* p. 8.

(*b*) *Two Subtracted.* Onondaga: 8, *teg-ueron,* from *tegni,* "2"; Crow: 8, *nupa-pik,* from *nupa,* "2," and *pirake,* "10"; Kwakiutl: *matl-gwanatl,* from *matl,* "2."

(*c*) *Three Subtracted.* This occurs but rarely. Pawnee is noteworthy. In it, 17, 18, 19 are *tauit-kaki, pitkus-kaki, usku-kaki*, from *usku,* "1," *pitkus,* "2," *tauit,* "3," and *kaki,* "less."

(*d*) *Ten Subtracted.* This is quite common among some of the California families, where the odd tens, 30, 50, 70, 90 are expressed as 40, 60, 80, 100 combined with ten, with the thought "60 lacking 10." In some cases however the thought may be "(40) and 10 on the way to 60," a variation of the additive principle not uncommon among these tribes [1]

4. **Multiplicative Principle.** This principle, like the additive, is of universal use in Indian as well as in all other languages for the formation of higher numerals. It seems to begin in the Indian languages with the expression of 4 as "2×2." It is constantly used in the formation of "secondary bases" in the ternary, quaternary, quinary, decimal, and vigesimal systems. Our chief interest in studying this principle is in the question of precedence. A much more decisive law of precedence than in the case of the additive principle is found. About 200 languages afford data for the conclusion that the marked tendency in the formation of higher numerals by multiplication is for the lesser number to precede the greater. This coincides with both the oral and written English forms. In every group the type $L \times G$ predominates over the type $G \times L$ to a marked degree. For the formation of the tens, forms of the type 2×10 predominate over the type 10×2 in the ratio 5 : 1; for the hundreds, 2 : 1; for the thousands, 2 : 1; for others (e. g., $8 = 2 \times 4$), 17 : 1. Or altogether a predominance of $L \times G$ 4 : 1. Almost all the instances of $G \times L$ are found in the languages about the Gulf of Mexico and in the single but large Siouan family of the plains. If the Siouan languages alone were left out of consideration the predominance of $L \times G$ would be about 8 : 1. In only five languages do we find $L \times G$ and $G \times L$ both occurring in the same system, a decided contrast to the results noted in the study of the additive principle.

5. **Duplicative Principle.** A striking feature of the Indian numeral systems is the frequency of occurrence of a duplicative or pairing principle. In some instances 6 is expressed as "2×3," "again 3," "3, 3," "threes" and similarly for 4, 8, 10, and even 12. The large number of natural pairs, such as the eyes, hands, arms, wings, etc., suggests that counting by pairs might be the course of evolution followed by some languages. The standard historians of mathematics make little reference to such a principle as of any importance in the numeral systems of primitive people.[2] We do find Hankel, however, making the rather surprising statement in discussing the number words of uncivilized people, that ten is never expressed as two times five, but always by a simple number word.[3] This is not the case in American Indian languages; e. g., Gabrieleno: *wehes-mahar,* from *wehe,* "2," and *mahar,* "5"; Serranos: *wor-maharte,* from *maharte,* "5,"

[1] See example of Maidu given under "Additive Principle."

[2] Cantor: *op. cit.*, p. 5. [3] Hankel: *op. cit.*, p. 20.

wor, "2"; Patwin: *pampa-semta*, from *eti-semta*, "5," *pampet*, "2." Examples of its use for 4, 6, 8, 12 are Kutchin: 6, *neckh-kiethei*, 8, *nakhei-etanna*, from *nackhai*, "2," *kiethai*, "3," *etanna*, "4"; Kansas: 8, *kiya-tuba*, from *tuba*, "4," *kiya*, "again"; Shoshone: 4, *what-sowit*, from *what*, "2"; Cehiga: 12, *cape-nanba*, from *cape*, "6," *nanba*, "2"; Chwachamaju: 8, *kom-tca*, from *mitca*, "4," *ko*, "2."

In languages of the same family and even dialects of the same language there is variation in the use of this principle. And many examples can be given where the principle is not used at all in languages closely related to those in which it is. One feature is rather surprising; namely, of the approximately 125 cases of the probable use of this principle, it is far more common in the formation of four, six, eight than in the case of ten, even though ten is represented so commonly by the two hands. Fifty instances of its use for the formation of "8" are found, thirty-five for "4" (although some of these are somewhat obscure), twenty-five for "6," only ten for "10" and two for "12." Several languages seem to use it regularly in the formation of the even numbers to ten.

6. **Divisive Principle.** Historians of mathematics agree on the rarity of the use of this principle in the formation of numerals the world over.[1] Only two or three possible instances have been found in Indian languages. Thus we have Unalit: 10, *kolin*, and the literal meaning of the word, "upper half of the body," Point Barrow: 10, *kodlin*, "upper part (of body)," and similarly in other Eskimo languages. One other example has been found, Pawnee: 5, *sihuks*, "hands half" from *ishu*, "hand," and *huks*, "half," i. e., "half of the two hands."

7. **Fractions.** Although many tribes had numeral systems of integers running into the thousands and even to millions,[2] very few of them had any idea of fractions. Where we do find such ideas they are of the most rudimentary sort. "One half" occurs most frequently, but only in about a dozen cases as far as noted, while examples of the other fractions are almost negligible.[3] It is worth mentioning that the few instances we find are all of "unit fractions."[4] Onondaga shows the best development of fractions, but how meager for a language whose numerals are given to one million. Its fractions are: $\frac{1}{2}$, *sat wachenonk*, meaning uncertain; $\frac{1}{3}$, *achen-na-degayagui*, from *achen*, "3," meaning "thrice divided"; $\frac{1}{4}$, *gayeri-degayagui*, from *gayeri*, "4," "four times divided."

8. **Notation.** As already mentioned, few symbols for number other than the spoken words are found. But on grave posts, buffalo robes, tattooing and other mnenomic pictographs there are a few pictured symbols.[5] The usual method used for indicating numbers is by the repetition of single strokes, i. e., the additive

[1] Fink: *op. cit.*, p. 8; Hankel: *op. cit.*, p. 21.

[2] As to actual use of these numbers, see below, "Limits in Use."

[3] Some ethnologists state that words for the simpler fractions were common, but that they have been recorded but rarely by students of Indian languages.

[4] The pedagogical suggestion is worth noting. The use of integers developed long before fractions, and when fractions are introduced the unit fractions, i. e., those which have unity for the numerator, are the first to appear. In the Ahmes papyrus, one of the earliest known mathematical documents (Egyptian), a table is given for reducing common fractions to unit fractions, which were regarded as the standard type. See Cajori: *History of Mathematics*, New York, 1894, p. 14; Ball: *Short History of Mathematics*, London, 1893, p. 3; Cantor, *op. cit.*, Chap. I.

[5] For reproductions of all those known in North America and discussion of them see articles by Mallery in 4th and 10th Ann. Rpts. Bur. Eth., Washington.

principle in its simplest form. Sometimes the strokes are arranged in rows of ten or every tenth stroke is made longer than the others. Instances of this kind are found for the expression of numbers as high as thirty. They are found among the more highly civilized Indians of the middle west, especially among the Dakotas. On the other hand the Comanches are said to have been "ignorant of the elements of figures, even of a perpendicular stroke for one."[1] Pure numeral notation is not always found. Frequently the number of objects is expressed by the repetition of the symbol for the object the desired number of times, especially in the case of men or tepees. Another method is by dots. A man's head with eight dots above it in one case means nine men, the head itself counting for one. Another picture gives a head over which are thirty black dots in three lines of ten each. This is said to mean thirty men, not thirty-one. Thus the usage is not fixed. The use of notches cut on sticks was frequent, not only in the middle west but in California and on Puget Sound in western Washington. At the San Gabriel mission in California every tenth notch was cut entirely across the stick instead of only in the corner.

(*To be continued.*)

SYNTHETIC PROJECTIVE GEOMETRY AS AN UNDERGRADUATE STUDY.

By W. H. BUSSEY, University of Minnesota.

Modern Geometry.—Synthetic projective geometry is called modern synthetic geometry because the theories and methods that make of its propositions a homogeneous and harmonious whole are due for the most part to men of a time near our own, namely Poncelet, Möbius, Steiner, Chasles, Von Staudt, Reye, Cremona, and others. Some of its theorems can be traced back to Euclid, Apollonius, and Pappus, but it is in the work of Pascal and Desargues that the germs of the modern theories are to be found. When he was only sixteen years old Pascal discovered his famous theorem about a hexagon inscribed in a conic and made it the basis of a treatise on conics which is unfortunately lost. It was never published but we know something of its contents and extent from a report made by Leibnitz who saw it in Paris. Two of the most important theorems of projective geometry go by the name of Desargues. One is that if any two triangles in a plane or in space are such that corresponding vertices are collinear in pairs with a fixed point, then the corresponding sides are concurrent in pairs with a fixed straight line. The other is that any transversal meets a conic and the pairs of opposite sides of any inscribed quadrangle in points of an involution. But unfortunately for the development of the subject the work of Pascal and Desargues was overshadowed by the geometry of their contemporary Descartes. Analytic geometry and the calculus, invented about thirty years later, became the

[1] Eakins, D. W.: in Schoolcraft's *Indian Tribes*, Philadelphia, 1851–60, Vol. 1, p. 416.